Assignment-3 Model Solutions

①. Given transition probability matrix :-

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 1/4 & 1/2 & 1/4 \\ 2 & 1/3 & 0 & 2/3 \\ 3 & 1/2 & 0 & 1/2 \end{array}$$

(I). $P(X_0=1, X_1=2) = P(X_1=2 \mid X_0=1).P(X_0=1)$

$$= \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$$

(II) $P(X_0=1, X_1=2, X_2=3) = P(X_2=3 \mid X_1=2, X_0=1).P(X_1=2, X_0=$

$$= P(X_2=3 \mid X_1=2).P(X_1=2 \mid X_0=1).P(X_0=$$

$$= \frac{\cancel{2}}{3} \times \frac{1}{\cancel{2}} \times \frac{1}{3} = \frac{1}{9}$$

②. (11.1) $\boxed{11}$

$$\begin{array}{c|ccc} & \text{Professional} & \text{Skilled} & \text{Unskilled} \\ \hline \text{Professional} & 0.8 & 0.1 & 0.1 \\ \text{Skilled} & 0.2 & 0.6 & 0.2 \\ \text{Unskilled} & 0.25 & 0.25 & 0.5 \end{array}$$

Let, Professional denoted by 1
Skilled " " 2
Unskilled " " 3

∴ Randomly chosen grandson of an unskilled laborer is a professional man $= P^2_{31} = P_{31}P_{11} + P_{32}P_{21} + P_{33}P_{31}$

$$= 0.25 \times 0.8 + 0.25 \times 0.2 + 0.5 \times 0.25$$

$$= 0.3750$$

11.1/19

Let, $D \to$ Double headed biased coin

$\sim D \to$ Fair coin

$H_n \to$ Head on n^{th} toss

$H_{n+1} \to$ Head on $(n+1)^{th}$ toss

$H_{n-1} \to$ Head on $(n-1)^{th}$ toss.

(a).

$$P(H_{n+1} | H_n) = \frac{P(H_{n+1} \cap H_n)}{P(H_n)} = \frac{P(H_n \cap H_{n+1} | D) \cdot P(D) + P(H_n \cap H_{n+1} | \sim D) P(}{P(H_n | D) P(D) + P(H_n | \sim D) \cdot P(\sim D)}$$

$$= \frac{1 \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}}{1 * \frac{1}{2} + \frac{1}{2} * \frac{1}{2}}$$

$$= \frac{\frac{1}{2} + \frac{1}{8}}{\frac{1}{2} + \frac{1}{4}} = \frac{\frac{5}{8}}{3/4} = \frac{5}{\cancel{8}_2} \times \frac{\cancel{4}}{3}$$

$$= \frac{5}{6}$$ Ans

(b).

	H	T
H	5/6	?
T	?	?

$$P(H_{n+1} | T_n) = \frac{P(H_{n+1} \cap T_n)}{P(T_n)} = \frac{P(H_{n+1} \cap T_n | D) P(D) + P(H_{n+1} \cap T_n | \sim D) P(\sim D)}{P(T_n | D) P(D) + P(T_n | \sim D) P(\sim D)}$$

$$= \frac{0 + \frac{1}{2} * \frac{1}{2} * \frac{1}{2}}{6 * \frac{1}{2} + \frac{1}{2} * \frac{1}{2}} = \frac{1}{8} \times 4 = \frac{1}{2}.$$

$$P(T_{n+1} | H_n) = \frac{P(T_{n+1} \cap H_n)}{P(H_n)} = \frac{P(T_{n+1} \cap H_n | D) P(D) + P(T_{n+1} \cap H_n | \sim D) P(\sim D)}{P(H_n | D) P(D) + P(H_n | \sim D) P(\sim D)}$$

$$= \frac{0 * \frac{1}{2} + \frac{1}{2} * \frac{1}{2} * \frac{1}{2}}{1 * \frac{1}{2} + \frac{1}{2} * \frac{1}{2}}$$

$$= \frac{1}{8} \div \frac{3}{4} = \frac{1}{\cancel{8}_2} \times \frac{\cancel{4}}{3} = 1/6$$

$$P(T_{n+1} | T_n) = \frac{P(T_{n+1} \cap T_n)}{P(T_n)} = \frac{0 + \frac{1}{8}}{1/4} = \frac{1}{8} \times 4 = \frac{1}{2}.$$

∴ Ans is

	H	T
H	5/6	1/6
T	1/2	1/2

ⓒ. $$P(H_{n+1} \mid H_n, H_{n-1}) = \frac{P(H_{n+1} \cap H_n \cap H_{n-1})}{P(H_n \cap H_{n-1})}$$

$$= \frac{P(H_{n+1} \cap H_n \cap H_{n-1} \mid D)\, P(D) + P(H_{n+1} \cap H_n \cap H_{n-1} \mid \sim D) P(\sim}{P(H_n \cap H_{n-1} \mid D) \cdot P(D) + P(H_n \cap H_{n-1} \mid \sim D) \cdot P(\sim D)}$$

$$= \frac{1 * \frac{1}{2} + \frac{1}{2} * \frac{1}{2} * \frac{1}{2} * \frac{1}{2}}{1 * \frac{1}{2} + \frac{1}{2} * \frac{1}{2} * \frac{1}{2}}$$

$$= \frac{\frac{1}{2} + \frac{1}{16}}{\frac{1}{2} + \frac{1}{8}} = \frac{9}{16} \div \frac{5}{8} = \frac{9}{\cancel{16}_{2}} \times \frac{\cancel{8}}{5} = \frac{9}{10}.$$

ⓓ. It is not a markov chain since ⓐ and ⓒ has different results.

(11.2/9)

$$P = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ 2 & 0 & 0 & 1/3 & 1/3 & 1/3 \\ 3 & 0 & 0 & 0 & 1/2 & 1/2 \\ 4 & 0 & 0 & 0 & 0 & 1 \\ 5 & 0 & 0 & 0 & 0 & 1 \end{array}$$

$$Q = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1/4 & 1/4 & 1/4 \\ 2 & 0 & 0 & 1/3 & 1/3 \\ 3 & 0 & 0 & 0 & \frac{1}{2} \\ 4 & 0 & 0 & 0 & 0 \end{array} \qquad R = \begin{array}{c|c} & 5 \\ \hline 1 & 1/4 \\ 2 & 1/3 \\ 3 & 1/2 \\ 4 & 1 \end{array}$$

$$N = (I - Q)^{-1}$$

t_i = expected # of steps taken before reaching absorbed state

$$\therefore\; t = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix} = N \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 1 & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & -1/2 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 25/12 \\ 11/6 \\ 3/2 \\ 1 \end{bmatrix}$$

$\boxed{11.2/13}$

(a) The transition matrix in canonical form:—

Timid Strategy

$$P = \begin{array}{c|ccccccc|cc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 0 \\ \hline 1 & 0 & .4 & 0 & 0 & 0 & 0 & 0 & 0 & .6 \\ 2 & .6 & 0 & .4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & .6 & 0 & .4 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & .6 & 0 & .4 & 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 & .6 & 0 & .4 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 & .0 & .6 & 0 & .4 & 0 & 0 \\ 7 & 0 & 0 & 0 & 0 & 0 & .6 & 0 & .4 & 0 \\ \hline 8 & 0 & 0 & 0 & 0 & 0 & 0 & .0 & 1 & .0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}$$

$$N = (I-Q)^{-1} = \left[\begin{array}{c|ccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 1 & 1 & -.4 & 0 & 0 & 0 & 0 & 0 \\ 2 & -.6 & 1 & -.4 & 0 & 0 & 0 & 0 \\ 3 & 0 & -.6 & 1 & -.4 & 0 & 0 & 0 \\ 4 & 0 & 0 & -.6 & 1 & -.4 & 0 & 0 \\ 5 & 0 & 0 & 0 & -.6 & 1 & -.4 & 0 \\ 6 & 0 & 0 & 0 & 0 & -.6 & 1 & -.4 \\ 7 & 0 & 0 & 0 & 0 & 0 & -.6 & 1 \end{array}\right]^{-1}$$

$$R = \begin{array}{c|cc} & 8 & 0 \\ \hline 1 & 0 & .6 \\ 2 & 0 & 0 \\ 3 & 0 & 0 \\ 4 & 0 & 0 \\ 5 & 0 & 0 \\ 6 & 0 & 0 \\ 7 & 0.4 & 0 \end{array}$$

$$B = NR = \begin{array}{c|cc} & 8 & 0 \\ \hline 1 & 0.02 & 0.97 \\ 2 & 0.04 & 0.94 \\ 3 & 0.09 & 0.89 \\ 4 & 0.16 & 0.83 \\ 5 & 0.26 & 0.73 \\ 6 & 0.42 & 0.57 \\ 7 & 0.65 & 0.34 \end{array}$$

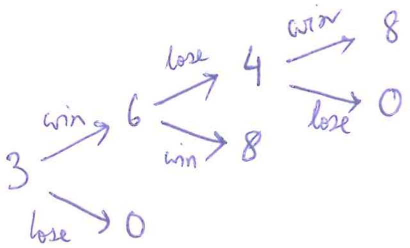
$$\therefore b_{38} = (NR)_{38} = 0.09$$

∴ Starting with 3 dollars, the probability of winning 8 dollars before losing all his money is 0.09

11.2
13

(b)

Bold strategy



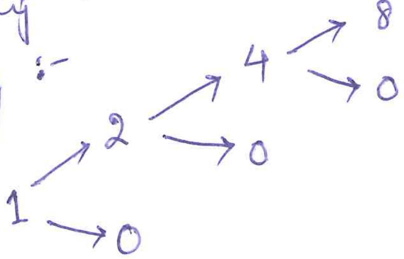
$$P = \begin{array}{c|c} & \begin{array}{ccccc} 3 & 4 & 6 & 0 & 8 \end{array} \\ \begin{array}{c} 3 \\ 4 \\ 6 \\ 0 \\ 8 \end{array} & \left[\begin{array}{ccc|cc} 0 & 0 & .4 & .6 & 0 \\ 0 & 0 & 0 & .6 & .4 \\ 0 & .6 & 0 & 0 & .4 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array}$$

$$B = \begin{array}{c|c} & \begin{array}{cc} 0 & 8 \end{array} \\ \begin{array}{c} 3 \\ 4 \\ 6 \end{array} & \left[\begin{array}{cc} 0.744 & 0.256 \\ 0.6 & 0.4 \\ 0.36 & 0.64 \end{array}\right] \end{array}$$

∴ Prob. of winning 8 dollars starting with 3 dollars will be 0.256

(c). Bold strategy gives better chance of getting out of jail.

** Note:- The above problem has been solved considering that Smith initially has 3 dollars. If the initial amount is considered as 1 dollar [as per some versions of the book], then the transition probability matrix of (b) will change according to the following:-



11.2/15

ⓐ. The re-arranged form of P is

$$\begin{array}{c|ccc|cc} & 3 & 4 & 5 & 1 & 2 \\ \hline 3 & 0 & .4 & 0 & .6 & 0 \\ 4 & .6 & 0 & .4 & 0 & 0 \\ 5 & 0 & .6 & 0 & 0 & .4 \\ \hline 1 & 0 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 \end{array}$$

ⓑ.

$$Q = \begin{array}{c|ccc} & 3 & 4 & 5 \\ \hline 3 & 0 & 0.4 & 0 \\ 4 & 0.6 & 0 & 0.4 \\ 5 & 0 & 0.6 & 0 \end{array} \qquad R = \begin{array}{c|cc} & 1 & 2 \\ \hline 3 & .6 & 0 \\ 4 & 0 & 0 \\ 5 & 0 & .4 \end{array}$$

$$N = (I - Q)^{-1} = \begin{bmatrix} 1 & -0.4 & 0 \\ -0.6 & 1 & -0.4 \\ 0 & -0.6 & 1 \end{bmatrix}^{-1}$$

$$\therefore\ t = Nc = \begin{array}{c} \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccc} 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & -0.4 & 0 \\ -0.6 & 1 & -0.4 \\ 0 & -0.6 & 1 \end{bmatrix}^{-1} \end{array} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.539 \\ 4.226 \\ 3.308 \end{bmatrix}$$

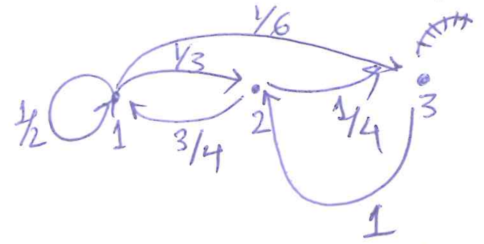
Absorption Probabilities

$$= B = NR = \begin{array}{c|ccc} & 3 & 4 & 5 \\ \hline 3 & 1.46 & 0.76 & 0.30 \\ 4 & 1.15 & 1.92 & 0.76 \\ 5 & 0.69 & 1.15 & 1.46 \end{array} \begin{array}{c|cc} & 1 & 2 \\ \hline 3 & .6 & 0 \\ 4 & 0 & 0 \\ 5 & 0 & .4 \end{array} = \begin{array}{c|cc} & 1 & 2 \\ \hline 3 & 0.877 & 0.12 \\ 4 & 0.69 & 0.30 \\ 5 & 0.41 & 0.58 \end{array}$$

ⓒ. At duece (state 4) expected duration of game is obtained from matrix $t = 4.226$.

Probability of B winning is obtained from matrix B = 0.30

(11.3/2) (a). The state transition diagram



For regular markov chain, we need to find 'n' such that $P_{ij}^{n} > 0$

One can see that in P^{2}, $P_{32}^{2} = 0$

i.e., starting from 3 the markov chain can't reach state 2 in 2 steps although rest all $P_{ij}^{2} > 0$

But for all i, j, $P_{ij}^{3} > 0$

[Remark : For smaller cases, one can directly observe / find n, otherwise one should go for finding matrix powers.]

(b). $P_{13}^{2} = P_{11} P_{13} + P_{12} P_{23} + P_{13} P_{33}$

$= \frac{1}{2} \cdot \frac{1}{6} + \frac{1}{3} \cdot \frac{1}{4} + \frac{1}{6} \cdot 0 = \frac{1}{6}$

(c) We need to solve $wP = w$
such that $\sum_{i=1}^{3} w_i = 1$

The equations are :-

$w_1 + w_2 + w_3 = 1$

$\frac{1}{2} w_1 + \frac{3}{4} w_2 + 0 w_3 = w_1$

$\frac{1}{3} w_1 + 0 w_2 + 1 w_3 = w_2$

$\frac{1}{6} w_1 + \frac{1}{4} w_2 + 0 w_3 = w_3$

Solving we get, $w_1 = \frac{1}{2}$, $w_2 = \frac{1}{3}$, $w_3 = \frac{1}{6}$.

11.3/3

ⓐ $a = 0$, or $b = 0$ or both $a = 0 = b$

ⓑ $a = 1 = b$ ~~or~~ ~~$0<b<1$~~

ⓒ $0<a<1$ ~~or~~ & $0<b<1$

or, $a = 1$ & $0<b<1$

or, $0<a<1$ & $b = 1$

11.3/5(c)

Solve, $wP = w$
such that $\sum w_i = 1$

Ans → $w = \left(\frac{2}{7}, \frac{3}{7}, \frac{2}{7}\right)$

11.3/6

With $a = b = 1$, $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$P^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $P^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$P^{2n} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $P^{2n+1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\Rightarrow P^n$ does not converge

$$A_n = \frac{I + P + P^2 + \cdots + P^n}{n+1}$$

Consider two cases, n; even and n: odd and show that in both cases A_n converges to the same matrix. $\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$

Each row of A_n converges to the same row vector a

12 : ~~4~~/8

See W. Feller,

$\overline{\underline{III}}$. 3 The MAIN LEMMA
[Page 76-77]

12·2/5

ⓐ. p : prob. of winning \$1 in a game
q : prob. of losing \$1 in a game

q_k : prob. of gambler's stake return to zero before reaching M if initial state was k

We need to find $q_0 = ?$

As, gambler is allowed to have negative amount

$$q_{-k} = 1 \ \forall k$$

Also, $q_0 = pq_1 + q \cdot q_{-1}$ [law of total probability]

$q_0 = pq_1 + q = pq_1 + (1-p) = 1 - p(1-q_1)$

$$\boxed{q_0 = 1 - p(1 - q_1)}$$

ⓑ. Solve 4(a) and 4(b)

This gives $p_0 = pp1 + \cancelto{0}{qp_{-1}}$

$= p(1 - q_1)$

Similarly, $q_M = q q_{m-1}$ [Need some arguments left deliberately for you to fill the gap]

Now, Required probability M before returning to 0

= (prob. of stake reaching zero) * with initial stake being (k-1) times exactly

(prob. of stake reaching M before returning to 0 with initial state again being M) *

(prob. of stake reaching 0 before returning to M with initial stake being M)

$= p(1-q_1)(1 - qq_{M-1})^{k-1}(q\,q_{M-1})$

③ . Given eq^n. of plane $3x - 4y + z = 2$ — ①

$N = (3, -4, 1)$ is normal to the plane.

A line passing through P and $\perp$r to the plane is given as $X = (1, 2, -1) + t(3, -4, 1)$

$= (1+3t, 2-4t, t-1)$

for intersection point $(1+3t, 2-4t, t-1)$ should also lie on ①

i.e., $3(1-3t) - 4(2-4t) + (t-1) = 2$

$\Rightarrow t = 4/13$

Intersection point $X = \left(\frac{25}{13}, \frac{10}{13}, -\frac{9}{13}\right)$

Ans.

④ . One way to construct such an example:- Construct 3 planes such that there is pairwise intersection between the two but not all three together.

⑤ . Given,

$$\begin{aligned} 2x + 5y + z &= 0 \\ 4x + ay + z &= 2 \\ y - z &= 3 \end{aligned} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{aligned} 2x + 5y + z &= 0 \quad -(1) \\ (a-10)y - z &= 2 \quad -(2) \\ y - z &= 3 \quad -(3) \end{aligned}$$

(1). If $a = 10$, coefficient of y becomes 0 and hence row 2 and 3 will be swapped to get 2nd pivot.

(2). If $a = 11$ then L.H.S. of both eq^n (2) and (3) will be same but R.H.S would be different leading to situation of no third-pivot as well as $0 \neq 0$