1) Let $X \sim \text{Hypergeometric}(b, r, k)$.

$\therefore$ PMF of $X = P_X(x) = \begin{cases} \dfrac{\binom{b}{x}\binom{r}{k-x}}{\binom{b+r}{k}}, & \text{for } x \in R_X \\ 0, & \text{o.w.} \end{cases}$

where R_X = range of X

$= \{\max(0, k-r), (\max(0, k-r)+1), \ldots, \min(k, b)\}$.

To prove that $P_X(x)$ is valid, we need to show that (i) $P_X(x) \geq 0$, $\forall x$.

(ii) $\sum_{x \in R_X} P_X(x) = 1$.

Proof for (i)

$\binom{b}{x}, \binom{r}{k-x}, \binom{b+r}{k} \in$ Set of non-negative integers.

[$\because$ they ~~are n~~ represent no. of ways some objects can be combined.]

$\therefore$ For any value of x, $\dfrac{\binom{b}{x}\binom{r}{k-x}}{\binom{b+r}{k}} \geq 0 \Rightarrow P_X(x) \geq 0$.

(Proved.)

Proof for (ii)

$$\sum_{x \in R_X} P_X(x) = \sum_{x \in R_X} \frac{\binom{b}{x}\binom{r}{k-x}}{\binom{b+r}{k}}$$

$$= \binom{b+r}{k}^{-1} \sum_{x \in R_X} \binom{b}{x}\binom{r}{k-x} \quad \cdots (e1)$$

By binomial theorem,

$$\sum_{k=0}^{b+r} \binom{b+r}{k} y^k = (1+y)^{b+r}$$

$$= (1+y)^b (1+y)^r$$

$$= \left(\sum_{i=0}^{b} \binom{b}{i} y^i\right)\left(\sum_{j=0}^{r} \binom{r}{j} y^j\right)$$

[Again by binomial theorem.

$$= \sum_{k=0}^{b+r} \left(\sum_{x=0}^{k} \binom{b}{x} y^x \cdot \binom{r}{k-x} y^{k-x}\right)$$

[∵ x replaces i ; $(k-x)$ replaces j ; All the bounds

<contd.>

$$= \sum_{k=0}^{b+r} \left(\sum_{x=0}^{k} \binom{b}{x}\binom{r}{k-x} y^k \right)$$

$$= \sum_{k=0}^{b+r} \left(\sum_{x=0}^{k} \binom{b}{x}\binom{r}{k-x} \right) y^k$$

$$\Rightarrow \sum_{k=0}^{b+r} \underbrace{\binom{b+r}{k}} y^k = \sum_{k=0}^{b+r} \underbrace{\left(\sum_{x=0}^{k} \binom{b}{x}\binom{r}{k-x} \right)} y^k$$

$$\Rightarrow \binom{b+r}{k} = \sum_{\substack{\cancel{k=0} \\ x=0}}^{k} \binom{b}{x}\binom{r}{k-x} \quad \cdots (e2)$$

<u>Note:</u> Equation (e2) is known as <u>Vandermonde's identity</u> when $m, n, r \in \mathbb{N}_0 = \{0, 1, 2, 3, \cdots\}$. You may check wikipedia for more interesting ways to prove this identity.

$\therefore$ For $\quad 0 \leq x \leq k,$

$$\underset{(0\leq X\leq k)}{P_X(x)} = \binom{b+r}{k}^{-1} \binom{b+r}{k} \quad \Big] \text{ by (e1), (e2)}$$

$$\Rightarrow \underset{(0\leq X\leq k)}{P(x)} = 1. \quad \cdots (e3)$$

For $\quad x < 0,$

$$\underset{(X<0)}{P(x)} = 0. \quad \cdots (e4) \quad \Big[\because \binom{b}{x} \text{ in (e1) will always be zero.}$$

For $\quad x > k,$

$$\underset{(x>k)}{P(x)} = 0. \quad \cdots (e5) \quad \Big[\because \binom{r}{k-x} \text{ in (e1) will be } \binom{r}{\text{some -ve value}} = 0.$$

$$\therefore \sum_{x \in R_X} P_X(x) = \sum_{X<0} P_X(x) + \sum_{0\leq X\leq k} P_X(x) + \sum_{X>k} P_X(x)$$

$$= 0 + 1 + 0 \quad \Big[\text{ By (e3), (e4), (e5)}$$

$$= 1. \text{ (Proved.)}$$

6. The Matching Problem

Sol^n: We previously found that

Please check "The Matching Problem" discussed in the HOS Book on page no 100 [sec2.1.5: Prob 7]

$$P(X_N=0) = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots (-1)^N \frac{1}{N!} \quad \text{for } N = 1, 2, \ldots$$

Using this we will find $P(X_N = k)$ for all $k \in \{0, 1, \ldots, N\}$

Let us first calculate $P(X_N = 1)$.

This is the probability that exactly one person receives his/her hat. We can fix this person. Therefore, the rest of the people ($N-1$ people) won't receive their own hats.

So, there are N ways to choose the person who gets his/her own hat.

The probability that the chosen person gets his/her hat is equal to $\frac{1}{N}$.

The probability that none of the other $N-1$ people receive their own hats is $P(X_{N-1}=0)$.

So, we have

$$P(X_N = 1) = N \cdot \frac{1}{N} \cdot P(X_{N-1}=0)$$
$$= P(X_{N-1}=0) = a_{N-1} \text{ (say)} \quad \text{---(i)}$$

Similarly, for calculating $P(X_N = 2)$, there are $\binom{N}{2}$ ways to choose two people who get their own hats.

$\frac{1}{N} \cdot \frac{1}{N-1}$ is the probability that the two chosen people receive their own hats and $P(X_{N-2}=0)$ is the probability that none of the other $N-2$ people receive their own hats.

$$P(X_N = 2) = \binom{N}{2} \cdot \frac{1}{N} \cdot \frac{1}{N-1} \cdot P(X_{N-2}=0) \quad \text{---(ii)}$$
$$= \frac{1}{2} P(X_{N-2}=0) = \frac{1}{2} a_{N-2}$$

In general we have,

$$P(X_N = k) = \binom{N}{k} \cdot \frac{1}{N} \cdot \frac{1}{N-1} \cdots \frac{1}{N-k+1} \cdot P(X_{N-k} = 0)$$

$$= \frac{1}{k!} P(X_{N-k} = 0) = \frac{1}{k!} a_{N-k} \text{ for } k = 0, 1, 2, \ldots, N$$

Solⁿ 9. Since $X \sim \text{Geometric}(p)$, we have:

$$P_X(k) = (1-p)^{k-1} p \text{ for } k = 1, 2, \ldots$$

Thus, $$P(X > m) = \sum_{k=m+1}^{\infty} (1-p)^{k-1} p$$

$$= \sum_{k=0}^{\infty} (1-p)^{k+m} p$$

$$= p(1-p)^m \sum_{k=0}^{\infty} (1-p)^k$$

$$= p(1-p)^m \frac{1}{1-(1-p)}$$

$$= (1-p)^m$$

Similarly, $P(X > m+l) = (1-p)^{m+l}$

Therefore, $$P(X > m+l \mid X > m) = \frac{P((X > m+l) \text{ and } P(X > m))}{P(X > m)}$$

$$= \frac{P(X > m+l)}{P(X > m)}$$

$$= \frac{(1-p)^{m+l}}{(1-p)^m}$$

$$= (1-p)^l$$

$$= P(X > l)$$

Proved

Sol^n: 11

(a). Time interval $= 4 \times 60$ mins $= 240$ mins

$\therefore \lambda = 240 \times \frac{1}{30} = 8$ (for a weekend)

Thus, $X \sim \text{Poisson}(\lambda = 8)$

Since no emails received in the interval

$$\therefore P(k=0) = \frac{e^{-\lambda}\lambda^{k}}{k!} = \frac{e^{-\lambda}\lambda^{0}}{0!}$$
$$= e^{-\lambda} = e^{-8}$$

(b). Let D be the event that a weekday is chosen and let E be the event that a saturday or sunday is chosen.

Then: $P(D) = \frac{5}{7}$ and $P(E) = \frac{2}{7}$

Let, A be the event that I receive no emails during the chosen interval then:

$$P(A|D) = e^{-\lambda_1} = e^{-\frac{1}{6}\cdot 60} = e^{-10}$$
$$P(A|E) = e^{-\lambda_2} = e^{-\frac{1}{30}\cdot 60} = e^{-2}$$

Therefore, $$P(D|A) = \frac{P(A|D)\cdot P(D)}{P(A)} = \frac{e^{-10}\cdot\frac{5}{7}}{P(A|D)P(D)+P(A|E)P(E)}$$
$$= \frac{\frac{5}{7}\cdot e^{-10}}{\frac{5}{7}\cdot e^{-10} + \frac{2}{7}e^{-2}}$$

Ans

Sol^n 21 Coupon collector's problem :-

(a) Let X denote the (random) number of coupons that we need to purchase in order to complete our collection. We can write $X = X_0 + X_1 + X_2 + \cdots + X_{N-1}$, where for any $i = 0, 1, 2, \ldots, N-1$, X_i denotes the additional number of coupons that we need to purchase to pass from ~~i~~ to $i+1$ different types of coupons in our collection.

Thus we have $X_0 = 1$ since the first coupon is always a new one.

Also, when i distinct types of coupons have been collected, a new coupon purchased will be of a distinct type with probability equal to $\frac{N-i}{N}$, i.e., X_i will be a geometric random variable with success probability of $\frac{N-i}{N}$.

Thus we can write $X = X_0 + X_1 + \cdots + X_{N-1}$ where

$$X_i \sim \text{Geometric}\left(\frac{N-i}{N}\right)$$

(b). By linearity of expectation, we have,

$$\begin{aligned} EX &= EX_0 + EX_1 + \cdots + EX_{N-1} \\ &= 1 + \frac{N}{N-1} + \frac{N}{N-2} + \cdots + \frac{N}{1} \\ &= N\left(1 + \frac{1}{2} + \cdots + \frac{1}{N-1} + \frac{1}{N}\right) \end{aligned}$$

Ans.

22) (a)

$R_X = \{1, 2, 4, 8, \cdots\}$.

$P_X(1) = \frac{1}{2}$.

$P_X(2) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} = \frac{1}{2^2}$.

$P_X(4) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8} = \frac{1}{2^3}$.

$\vdots$

$P_X(2^{k-1}) = \frac{1}{2^k}$ for $k \in \{1, 2, 3, \cdots\} = \mathbb{N} \cdots (e1)$

$\therefore EX = \sum_{x \in R_X} x \cdot P_X(x)$

$= \cancel{1 \cdot \frac{1}{2} + \cdots} \sum_{k \in \mathbb{N}} \left(2^{k-1} \cdot \frac{1}{2^k}\right)$ [by (e1)

$= \sum_{k \in \mathbb{N}} \frac{1}{2}$

$= \frac{\infty}{2}$ [$\because \mathbb{N}$ is an infinite set.

$= \infty$. (Ans.)

22. (b)

$$P(X>65) = P_X(128) + P_X(256) + \cdots$$

$$= P_X(2^{8-1}) + P_X(2^{9-1}) + \cdots$$

$$= \frac{1}{2^8} + \frac{1}{2^9} + \cdots \quad \Big] \text{ by (e1)}$$

$$= \frac{1}{2^8}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots\right)$$

$$= \frac{1}{2^8} \cdot \frac{1}{\left(1-\frac{1}{2}\right)} = \frac{1}{2^8} \cdot 2 = \frac{1}{2^7}$$

$$= \frac{1}{128} \quad \text{(Ans.)}$$

22. (c)

$$P_Y(2^{k-1}) = \begin{cases} \frac{1}{2^k}, & \text{for } k = 1, 2, \cdots, 30 \quad \Big[\text{like } X \text{ in (e1)} \\ \frac{1}{2^{30}}, & \text{for } k = 31 \\ 0, & \text{o.w.} \end{cases}$$

$$\therefore EY = \sum_{y \in R_Y} y \cdot P_Y(y) = \left(1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{2^2} + \cdots \cancel{+ 30 \cdot \frac{1}{}} \cdots + 2^{29} \cdot \frac{1}{2^{30}}\right) + 2^{30} \cdot \frac{1}{2^{30}}$$

<P.T.O.>

22. (c) <contd.>

$$\Rightarrow EY = \left(\frac{1}{2} + \frac{1}{2} + \cdots \text{ 30 times}\right) + 1$$

$$= \frac{30}{2} + 1$$

$$= 15 + 1$$

$$= 16. \quad \text{(Ans.)}$$

25) (a)

Case $m = 1$

$$P(X \geq m) = P(X \geq 1) = 0.4 + 0.3 + 0.3 + 0$$

$$= \cancel{0.6}\ 1 \geq \frac{1}{2}.$$

$$P(X \leq m) = P(X \leq 1) = 0.4 + 0$$

$$= 0.4 < \frac{1}{2}.$$

Reached a contradiction.

Case $m = 2$

$$P(X \geq m) = P(X \geq 2)$$

$$= 0.3 + 0.3 + 0$$

$$= 0.6 \geq \frac{1}{2}$$

$$P(X \leq m) = P(X \leq 2)$$

$$= 0.4 + 0.3 + 0$$

$$= 0.7 \geq \frac{1}{2}.$$

$\therefore$ $m = 2$ is valid.

Case $m = 3$ $\quad P(X \geq m) = P(X \geq 3) = 0.3 + 0 = 0.3 < \frac{1}{2}.$

Reached a contradiction.

$\therefore$ Median of $X = m = 2$. (Ans.)

25. (b)

PMF of X

$$= P_X(k) = \begin{cases} \frac{1}{6}, & \text{for } k = 1, 2, \cdots, 6 \\ 0, & \text{o.w.} \end{cases}$$

Case $m = 1$

$$P(X \le m) = P(X \le 1) = \frac{1}{6} + 0 = \frac{1}{6} < \frac{1}{2}.$$

Contradiction.

Case $m = 2$

$$P(X \le m) = P(X \le 2) = \frac{1}{6} + \frac{1}{6} + 0 = \frac{1}{3} < \frac{1}{2}.$$

contradiction.

Case $m = 3$

$$P(X \le m) = P(X \le 3) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + 0$$

$$= \frac{1}{2} \geq \frac{1}{2}.$$

$$P(X \geq m) = P(X \geq 3) = \cdots = \frac{1}{2} \geq \frac{1}{2}.$$

$\therefore$ $m = 3$ is valid.

Case $m = 4$

$$P(X \le 4) = \frac{4}{6} = \frac{2}{3} \geq \frac{1}{2}.$$

$$P(X \geq 4) = \frac{3}{6} = \frac{1}{2} \geq \frac{1}{2}.$$

$\therefore$ $m = 4$ is valid.

<P.T.O.>

25. (b) <contd.>

Case $m=5$

$$P(X \le 5) = \frac{5}{6} \geq \frac{1}{2}.$$

$$P(X \geq 5) = \frac{2}{6} = \frac{1}{3} < \frac{1}{2}.$$ Contradiction.

Case $m=6$

$$P(X \le 6) = \frac{6}{6} = 1 \geq \frac{1}{2}.$$

$$P(X \geq 6) = \frac{1}{6} < \frac{1}{2}.$$ Contradiction.

$\therefore$ Medians of X ~~is~~ are $3, 4$. (Ans.)

25. (c) $X \sim \text{Geometric}(p)$, where $0 < p < 1$.

$\Rightarrow$ PMF of X $= P_X(k) = (1-p)^{k-1} p$

$= q^{k-1} \cdot p$ ... (e0) [Let $q = (1-p)$. ... (e3)

Let m is a median of X.

$\therefore$ $P(X \geq m) \geq \frac{1}{2}$ ... (e1)

and $P(X \le m) \geq \frac{1}{2}$. ... (e2)

<P.T.O.>

25. (c) <contd>

$$P(X \le m) = \sum_{k=1}^{\lfloor m \rfloor} q^{k-1} p$$

By (e0).
$\lfloor m \rfloor$ is taken ~~in case~~ to tighten the upper bound.

$$= p \cdot \sum_{k=1}^{\lfloor m \rfloor} q^{k-1}$$

$$= p \cdot \frac{\left(1 - q^{\lfloor m \rfloor - 1 + 1}\right)}{(1-q)}$$

$$= p \cdot \frac{\left(1 - q^{\lfloor m \rfloor}\right)}{(1-q)}$$

$$\Rightarrow P(X \le m) = \left(1 - q^{\lfloor m \rfloor}\right) \quad \ldots (e4)$$

By (e3) and $\because p > 0$.

$$\left(1 - q^{\lfloor m \rfloor}\right) \ge \frac{1}{2}$$

By (e2) and (e4)

$$\Rightarrow q^{\lfloor m \rfloor} \le \frac{1}{2}$$

$$\Rightarrow \lg q^{\lfloor m \rfloor} \le \lg \frac{1}{2} = -1$$

$$\Rightarrow \lfloor m \rfloor \lg q \le -1$$

$$\Rightarrow -\lfloor m \rfloor \lg q \ge 1$$

<P.T.O.>

25.(c) <contd.>

$$\Rightarrow \lfloor m \rfloor \lg q^{-1} \geq 1$$

$$\Rightarrow \lfloor m \rfloor \geq \frac{1}{\lg\left(\frac{1}{q}\right)} \quad \cdots (e5)$$

Now, $P(X \geq m) = \sum_{k=\lceil m \rceil}^{\infty} q^{k-1} p$ [By (e0). $\lceil m \rceil$ is used to tighten the lower bound.

$$= q^{\lceil m \rceil - 1} p + q^{\lceil m \rceil + 1 - 1} p + q^{\lceil m \rceil + 2 - 1} p + \cdots$$

$$= q^{\lceil m \rceil - 1} \cdot p \left(1 + q + q^2 + \cdots \infty\right)$$

$$= q^{\lceil m \rceil - 1} \cdot p \cdot \frac{1}{1-q} \quad \left[\because q < 1.\right.$$

$$\Rightarrow P(X \geq m) = q^{\lceil m \rceil - 1} \quad \cdots (e6) \quad \left[\text{By } (e3) \text{ and } \because p > 0.\right.$$

$$q^{\lceil m \rceil - 1} \geq \frac{1}{2} \quad \left[\text{By } (e1) \text{ and } (e6)\right.$$

$$\Rightarrow (\lceil m \rceil - 1) \lg q \geq -1$$

$$\Rightarrow (\lceil m \rceil - 1) \lg \frac{1}{q} \leq 1$$

<P.T.O.>

25. (c) <contd.>

$$\Rightarrow \lceil m \rceil - 1 \leq \frac{1}{\lg \frac{1}{q}}$$

$$\Rightarrow \lceil m \rceil \leq \frac{1}{\lg \frac{1}{q}} + 1. \quad \cdots (e7)$$

$\therefore$ Any $m \in R_X$ that satisfies

$$\lfloor m \rfloor \geq \frac{1}{\lg\left(\frac{1}{q}\right)} \quad \left[\text{By } (e5)\right.$$

and $$\lceil m \rceil \leq \frac{1}{\lg\left(\frac{1}{q}\right)} + 1 \quad \left[\text{By } (e7)\right.$$

is a median of X. (Ans.)

3)

Case 3.1 : Poisson distribution.

Ref: Wikipedia article on 'Infinite divisibility (probabiliti

Case 3.2 Binomial distribution Binomial (n, p)

where

n = total number of nodes,

p = probability of a node being idle.

Case 3.3 : Hypergeometric distribution.