Q1. Prove that set of Rational numbers Q is a countable set

Solution. A set is countable either if it is finite or it is infinite and we can find a one-to-one correspondence between the elements of the set and the set of natural numbers.
By the following tabular method, each and every fraction can be formed just by following the arrow and starting at $1/1$.

	1	2	3	4	5	6	7 ...
1	1/1	1/2	1/3	1/4	1/5	1/6	1/7
2	2/1	2/2	2/3	2/4	2/5	2/6	2/7
3	3/1	3/2	3/3	3/4	3/5	3/6	3/7
4	4/1	4/2	4/3	4/4	4/5	4/6	4/7
5	5/1	5/2	5/3	5/4	5/5	5/6	5/7
6	6/1	6/2	6/3	6/4	6/5	6/6	6/7
⋮							

All possible fractions will be in the list. For example x/y will be in the table at the intersection of the x^{th} row and y^{th} column. Equivalent fractions are skipped (shown here for few examples by putting a cross mark in the box).

Q(3).

(17).

We have $P(A) = P(B)$

$P(C) = 2P(D)$

$P(A \cup C) = 0.6$

$\therefore P(A) + P(C) - P(A \cap C) = 0.6$

$\Rightarrow P(A) + P(C) = 0.6$ — (i) $\left[\because \text{Only one of team A or C can win} \therefore P(A \cap C) = 0\right]$

Also, $P(A) + P(B) + P(C) + P(D) = 1$

$\Rightarrow P(B) + P(D) = 0.4$

$\Rightarrow P(A) + \frac{1}{2}P(C) = 0.4$

$\Rightarrow 2P(A) + P(C) = 0.8$ — (ii)

From (i) & (ii) by substitution, we get,

$P(A) = P(B) = P(D) = 0.2$

and $P(C) = 0.4$

Q(3). (25). Let C be the event that a random student lives on campus and A be the event that he/she gets an A.

We have,

$$P(A) \approx \frac{120}{600} = 1/5$$

$$P(C) \approx \frac{200}{600} = \frac{1}{3}$$

$$P(A \cap C^c) \approx \frac{80}{600} = \frac{2}{15}$$

$$P(A \cap C) = P(A) - P(A \cap C^c)$$

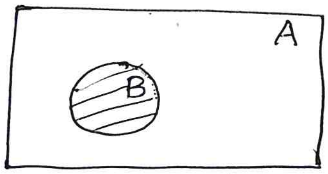
$$= \frac{1}{5} - \frac{2}{15} = \frac{3-2}{15} = \frac{1}{15}$$

Therefore, $P(A \cap C) = \frac{1}{15} = \frac{1}{5} \cdot \frac{1}{3} = P(A) \cdot P(C)$

Thus A and C are independent events.

2. a) P.T. $B \subset A \Rightarrow \begin{cases} P(B) \leq P(A) & \cdots (i) \\ P(A-B) = P(A) - P(B). & \cdots (ii) \end{cases}$

$A = (A-B) \cup B$ [$\because B \subset A$.



$\Rightarrow P(A) = P((A-B) \cup B)$

$= P(A-B) + P(B)$ [$\because$ $(A-B)$ and B are disjoint sets.

$\Rightarrow P(A-B) = P(A) - P(B). \cdots (ii)$. (Proved.)

From (ii),

$P(B) = P(A) - P(A-B)$

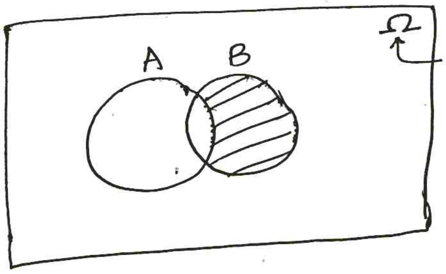
$\Rightarrow P(B) \leq P(A)$. [$\because P(A-B) \geq 0$.

$\cdots (i)$. (Proved.)

2.b) A and B are independent events

$\Rightarrow P(A \cap B) = P(A) \cdot P(B) \quad \ldots (i)$

P.T. $P(A^c \cap B) = P(A^c) \cdot P(B)$.



$B = (A \cap B) \cup (A^c \cap B)$

$\Rightarrow P(B) = P((A \cap B) \cup (A^c \cap B))$

$= P(A \cap B) + P(A^c \cap B)$ [$\because (A \cap B), (A^c \cap B)$ are disjoint sets.]

$\Rightarrow P(A^c \cap B) = P(B) - P(A \cap B)$

$= P(B) - P(A) \cdot P(B)$ [by (i)]

$= (1 - P(A)) \cdot P(B)$

$\Rightarrow P(A^c \cap B) = P(A^c) \cdot P(B)$. [$\because P(A^c) = 1 - P(A)$.]

(Proved.)

3. Chap 1. 10) a.

P.T. $A \xleftrightarrow{1:1} C$.

Let us have a mapping $f : A \mapsto C$

defined as

$f(B)$ where $B \subset \mathbb{N}$ i.e. $B \in A = 2^{\mathbb{N}}$

= a binary sequence of the length $|\mathbb{N}|$ i.e. infinite length where

$$k^{th} \text{ bit} = \begin{cases} 1, & \text{if } k \in B \\ 0, & \text{otherwise} \end{cases},$$

and $k \in \mathbb{N}$.

$f(\cdot)$ is a bijection

$\Rightarrow A \xleftrightarrow{1:1} C$. (Proved.)

3. Chap 1. 10) b.

P.T. $C \xleftrightarrow{1:1} [0, 1]$.

If we prepend a binary pt. to any element $c \in C$

then '$0.c$' $\in [0, 1]$.

But for some $c \in C$, they map to the same element in $[0, 1]$. E.g. '$0111\ldots$' and '$1000\ldots$'.

$0.1000\ldots = \frac{1}{2}$.

$0.0111\ldots = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$ (P.T.O.)

3. Chap. 1. 10) b.

$$0.1000\cdots = \frac{1}{2}.$$

$$0.0111\cdots = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

$$= \frac{\frac{1}{4}}{1-\frac{1}{2}}$$ [Using infinite geometric series sum formula.

$$= \frac{1}{2}.$$

~~Pu~~ Let us put all ~~the~~ such problem-creating sequences in a set $X = (\cdots, 1000\cdots, 0111\cdots, \cdots)$.

So, $X \subset C$.

~~Th~~ Similarly, put all such problem-creating ~~sequences~~ fractions (to whom the problem-creating sequences map) in a set $Y = (\frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8} \cdots)$.

[~~Sue~~ '<u>Note:</u> Such fractions have denominator as 2's power.

So, $Y \subset [0, 1]$.

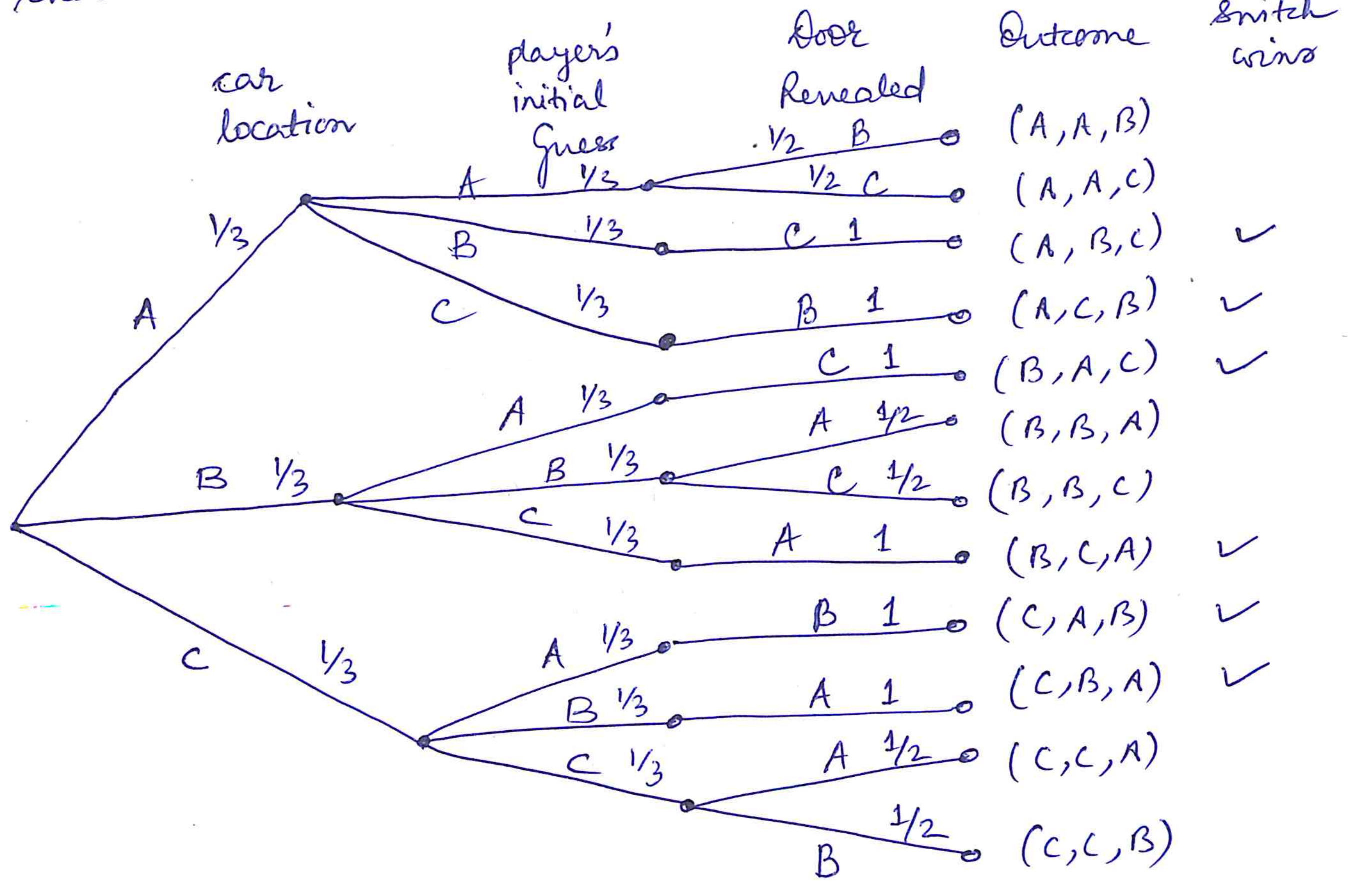
Now, define a bijection $g : C \longmapsto [0, 1]$ as

$$g(c) = \begin{cases} 0.c, & \text{if } c \in (C \setminus X). \\ n^{th} \text{ element in } Y, & \text{if } c \text{ is the } n^{th} \text{ element in } X. \end{cases}$$

(Proved.)

[<u>Note:</u> '111⋯' maps to '1' because '0.111⋯' $= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1.$]

Q(3).
(33). We can represent the Monty Hall Problem with the following tree diagram with the outcomes labeled for each path from root to leaf. For example, outcome (A, A, B) corresponds to the car being behind door A, the player initially choosing door A, and Monty (Host) revealing the goat behind door B. The outcomes in the event where the player wins by switching are denoted with a check mark.



The probability of an outcome is equal to the product of the edge-probabilities on the path from the root to that outcome. Therefore, $P(A, A, B) = \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{18}$.

∴ Probability of the event that the player wins by switching

$= Pr[\text{switching wins}] = Pr[(A, B, C)] + Pr[(A, C, B)] + Pr[(B, A, C)] + Pr[(B, C, A)] + Pr[(C, A, B)] + Pr[(C, B, A)]$

$= \frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9} = \frac{6}{9} = \frac{2}{3}$

∴ $Pr[\text{not switching wins}] = 1 - Pr[\text{switching wins}] = 1 - \frac{2}{3} = \frac{1}{3}$

∴ Switching is a better choice.

(3)
(3b). Let, A_n: The first n coin tosses result in heads.

H_{n+1}: The $(n+1)^{th}$ coin toss results in head.

C: The regular coin has been selected.

$$P(H_{n+1} \mid A_n) = P(H_{n+1} \mid C, A_n)\, P(C \mid A_n) + P(H_{n+1} \mid C^c, A_n)\, P(C^c \mid A_n)$$

Given, C (or C^c), A_n and H_{n+1} are independent

Thus, $P(H_{n+1} \mid A_n) = P(H_{n+1} \mid C)\, P(C \mid A_n) + P(H_{n+1} \mid C^c)\, P(C^c \mid A_n)$

$$P(C \mid A_n) = \frac{P(A_n \mid C) \cdot P(C)}{P(A_n)}$$

$$= \frac{P(A_n \mid C) \cdot P(C)}{P(A_n \mid C) \cdot P(C) + P(A_n \mid C^c) \cdot P(C^c)}$$

$$= \frac{\left(\frac{1}{2}\right)^n \cdot \frac{1}{2}}{\frac{1}{2}^n \cdot \frac{1}{2} + \frac{1}{2}} = \frac{1}{2^n + 1}$$

$$P(C^c \mid A_n) = 1 - P(C \mid A_n) = \frac{2^n}{2^n + 1}$$

Thus, $P(H_{n+1} \mid A_n) = \frac{1}{2} \cdot \frac{1}{2^n+1} + 1 \cdot \frac{2^n}{2^n+1}$

$$= \frac{2^{n+1} + 1}{2^{n+1} + 2}$$

Q(3)

(37). The sample space has 2^n elements.

$S = \{ (G, G, \cdots, G), (G, G, \cdots, B), \cdots, (B, B, \cdots, B) \}$

Let, A be the event that all the children are girls,

then $A = \{ (G, G, \cdots, G) \}$

thus, $P(A) = \frac{1}{2^n}$.

Let, B be the event that at least one child is girl, then $B = S - \{ (B, B, \cdots, B \}$

$|B| = 2^n - 1$

$P(B) = \frac{2^n - 1}{2^n}$.

Then, $A \cap B = A$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)} = \frac{\frac{1}{2^n}}{\frac{2^n - 1}{2^n}}$$

$$= \frac{1}{2^n - 1}.$$

3. Chap 2. 13)

Two coins.

Let, event C_1 = coin 1 is chosen.

" C_2 = " 2 " " .

Randomly chosen $\Rightarrow P(C_1) = P(C_2) = \frac{1}{2}$. ... (e1)

$$P(H) = \begin{cases} \frac{1}{2} & \text{for coin 1} \\ \frac{1}{3} & \text{" " 2} \end{cases}$$

Tossed 5 times.

a) $P(H \geq 3) = ?$

$P(H \geq 3)$

$= P(H \geq 3, C_1) + P(H \geq 3, C_2)$ [Using the law of total probability.

$= P(H \geq 3 | C_1) \cdot P(C_1) + P(H \geq 3 | C_2) \cdot P(C_2)$

[by Bayes' Rule.

$= P(H \geq 3 | C_1) \cdot \frac{1}{2} + P(H \geq 3 | C_2) \cdot \frac{1}{2}$ [by (e1).

$$= \Big(\big(P(H=3 | C_1) + P(H=4 | C_1) + P(H=5 | C_1) \big) + \big(P(H=3 | C_2) + P(H=4 | C_2) + P(H=5 | C_2) \big) \Big) \cdot \frac{1}{2}$$

<P.T.O.>

⟨contd.⟩ 3. Chap 2. 13) a.

$$= \frac{1}{2} \left(\left(\binom{5}{3}\left(\frac{1}{2}\right)^3\left(1-\frac{1}{2}\right)^2 + \binom{5}{4}\left(\frac{1}{2}\right)^4\left(1-\frac{1}{2}\right)^1 + \binom{5}{5}\left(\frac{1}{2}\right)^5\left(1-\frac{1}{2}\right)^0 \right) + \left(\binom{5}{3}\left(\frac{1}{3}\right)^3\left(1-\frac{1}{3}\right)^2 + \binom{5}{4}\left(\frac{1}{3}\right)^4\left(1-\frac{1}{3}\right)^1 + \binom{5}{5}\left(\frac{1}{3}\right)^5\left(1-\frac{1}{3}\right)^0 \right) \right) \quad \cdots (e2)$$

(Ans.)

3. Chap 2. 13) b.

$$P(C_2 \mid H \geq 3)$$

$$= \cancel{P(H \geq 3)} \; \frac{P(H \geq 3 \mid C_2) \cdot P(C_2)}{P(H \geq 3)}$$

$$= \frac{\left(\binom{5}{3}\left(\frac{1}{3}\right)^3\left(1-\frac{1}{3}\right)^2 + \binom{5}{4}\left(\frac{1}{3}\right)^4\left(1-\frac{1}{3}\right)^1 + \binom{5}{5}\left(\frac{1}{3}\right)^5\left(1-\frac{1}{3}\right)^0 \right) \cdot \frac{1}{2}}{\frac{1}{2} \cdot \left(\left(\binom{5}{3}\left(\frac{1}{2}\right)^3\left(1-\frac{1}{2}\right)^2 + \binom{5}{4}\left(\frac{1}{2}\right)^4\left(1-\frac{1}{2}\right)^1 + \binom{5}{5}\left(\frac{1}{2}\right)^5\left(1-\frac{1}{2}\right)^0 \right) + \left(\binom{5}{3}\left(\frac{1}{3}\right)^3\left(1-\frac{1}{3}\right)^2 + \binom{5}{4}\left(\frac{1}{3}\right)^4\left(1-\frac{1}{3}\right)^1 + \binom{5}{5}\left(\frac{1}{3}\right)^5\left(1-\frac{1}{3}\right)^0 \right) \right)}$$

[by (e1) and (e2).

(Ans.)

Chap 2. 14)

Method I

Say, 3 groups $\{G_1, G_2, G_3\}$ of 5 members each.

P (Hannah (H) and Sarah (S) are in the same group)

$$= P\left(\left(H, S \in G_1\right) \cup \left(H, S \in G_2\right) \cup \left(H, S \in G_3\right)\right)$$

$$= P\left(H, S \in G_1\right) + P\left(H, S \in G_2\right) + P\left(H, S \in G_3\right)$$

[$\because$ $(H, S \in G_i)$ for $i = 1,2,3$ are disjoint events.

$$= \frac{\binom{13}{3}\binom{10}{5}}{\binom{15}{5}\binom{10}{5}} + \frac{\binom{13}{3}\binom{10}{5}}{\binom{15}{5}\binom{10}{5}} + \frac{\binom{13}{3}\binom{10}{5}}{\binom{15}{5}\binom{10}{5}}$$

$$= 3 \cdot \frac{\binom{13}{3}\binom{10}{5}}{\binom{15}{5}\binom{10}{5}}$$

$$= \frac{2}{7}.$$ (Ans.)

[$\binom{15}{5}\binom{10}{5}$ = Number of ways to make 3 5-member groups from 15 people.

$\binom{13}{3}\binom{10}{5}$ = Number of ways to form 3 5-member groups from 15 people when 2 persons are already allocated to a predefined group.

<P.T.O.>

Chap 2. 14)

Method II (Shortcut technique)

Let us assign Hannah to any one group. Then we ~~have to~~ are left with 14 people who are equally likely to be one of the rest 4 members in Hannah's group.

$\therefore P\left(\begin{array}{l}\text{Any } \cancel{\text{other person}} \text{ person other than Hannah} \\ \text{to be in Hannah's group}\end{array}\right)$

$= \frac{4}{14}$

$= \frac{2}{7}$

$\therefore P(\text{Sarah is in Hannah's group})$

$= \frac{2}{7}$. (Ans.)

(v)6

3. Chap 2. 15)

P(at least one value is observed more than once)

$$= 1 - P(\text{every outcome is unique})$$

$$= 1 - \frac{\#\{\text{Number of permutations of size 5 from } \{1,\dots,6\} \text{ without repetition of values}\}}{\text{Number of all possible outcomes}}$$

$$= 1 - \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{6 \cdot 6 \cdot 6 \cdot 6 \cdot 6}$$

$$= 1 - \frac{6!}{6^5}$$

$$= \frac{49}{54}.$$ (Ans.)