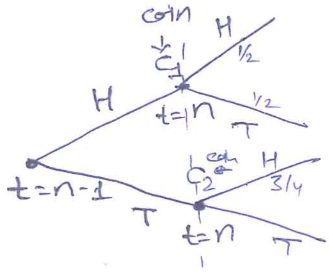
Quiz-1 Model Solution
CS 519 -

$\boxed{1.}$

Let
X_n = Toss-result at $t=n$

We have to find,
$$y_n = P(X_n = H)$$



Given,
For coin C_1, $P(H) = 1/2$
—— C_2 $P(H) = 3/4$

$$P(X_n = H) = P(X_n = H, X_{n-1} = H) + P(X_n = H, X_{n-1} = T) \quad \text{[Law of total prob.]}$$

$$= \underbrace{P(X_n = H \mid X_{n-1} = H)}_{\text{If } X_{n-1}=H\text{, then coin } C_1 \text{ was selected for toss at } t=n.} P(X_{n-1} = H) + \underbrace{P(X_n = H \mid X_{n-1} = T)}_{\text{similar argument but coin is } C_2} P(X_{n-1} = T)$$

$$= \frac{1}{2} \cdot P(X_{n-1} = H) + \frac{3}{4} \cdot P(X_{n-1} = T)$$

$$= \frac{1}{2} \cdot P(X_{n-1} = H) + \frac{3}{4}\left[1 - P(X_{n-1} = H)\right]$$

$$= \frac{1}{2} \cdot y_{n-1} + \frac{3}{4} - \frac{3}{4} y_{n-1}$$

i.e.
$$\boxed{y_n + \frac{1}{4} y_{n-1} = \frac{3}{4}} \rightarrow \text{Part a.}$$

Part (b) Solving the recurrence rel.^{n}

$$y_n = \frac{3}{4} - \frac{1}{4} y_{n-1}$$

$$y_n = \frac{3}{4} - \frac{1}{4}\left[\frac{3}{4} - \frac{1}{4} y_{n-2}\right]$$

$$= \frac{3}{4}\left[1 - \frac{1}{4}\right] + \frac{1}{4^2} y_{n-2}$$

$$= \frac{3}{4}\left[1 - \frac{1}{4}\right] + \frac{1}{4^2}\left[\frac{3}{4} - \frac{1}{4} y_{n-3}\right]$$

$$= \frac{3}{4}\left[1 - \frac{1}{4} + \frac{1}{4^2}\right] - \frac{1}{4^3} y_{n-3}$$

⋮

$$y_n = \frac{3}{4}\left[1 - 1/4 + 1/4^2 - \cdots + \left(\frac{-1}{4}\right)^{\cdots}\right] + \left(\frac{-1}{4}\right)^{n} y_0$$

but $y_0 = P(X_0 = H) = 1/2$ [As at $t=0$, coin C_1 is chosen for toss.]

$$y_n = 3/4 \left[\frac{1 - (-1/4)^n}{1 + 1/4}\right] + (-1/4)^n \cdot 1/2$$

$$= 3/5 - (-1/4)^n \left[3/5 - 1/2\right]$$

$$\boxed{y_n = 3/5 - \frac{1}{10}(-1/4)^n}$$

Q. 2. Case I: $P(A) = 0 \Rightarrow P(A)\cdot P(B) = 0$ for any event B.

Also ~~P(A∪B)~~ $P(B) \le P(A \cup B) \le \overset{0}{P(A)} + P(B)$ ~~often~~

ie $P(A \cup B) = P(B)$

$\Rightarrow P(A \cap B) = 0$ thus $\boxed{P(A \cap B) = P(A)\cdot P(B) = 0}$

Case II. $P(A) = 1 \Rightarrow P(A)\cdot P(B) = P(B)$ for any event B.

We will show $P(A \cap B) = P(B)$

$P(B) = P(A) + P(B) - 1 \le P(A \cap B)$

but $A \cap B \subseteq B \Rightarrow P(A \cap B) \le P(B)$

$\Rightarrow P(A \cap B) = P(B)$

ie $P(A \cap B) = P(B) = P(A)\cdot P(B)$.

Hence the result

Q. 3. HOS: Page 98. Ch 2. Solved problems. 5.